Clap-Clap: Formes modulaires, aspects théoriques et calculatoires
Roscoff, lundi 6 février 2017, 14h
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SMF of genus 2 and level 2

j.w. *G. van der Geer 2015
S. Grushevsky

Starting pt: Jonas Bergström, Carel Faber* } Cohomological computations and conjecture

Aim of today: prove **Thm:** The R^{ev}-module $M = \bigoplus_k M_{2,2k}(\Gamma_2[2])$ is generated by 15 modular forms G_{ij} $1 \leq i < j \leq 6$

$\Gamma_2 = Sp(4, \mathbb{Z})$

$\Gamma_2[2] = \{ g \in \Gamma_2 \ / \ g \equiv I_4 \mod 2 \}$

Main tool: $\Gamma_2 / \Gamma_2[2] \simeq Sp(4, \mathbb{Z}/2\mathbb{Z}) \simeq \mathfrak{S}_6$ — 11 irred repr bij with partition of 6

6	5+1	4+2	4+1+1	...	2+1+1+1+1	1+1+1+1+1+1
$s[6]$	$s[5,1]$		$s[4,1^2]$		$s[2,1^4]$	$s[1^6]$
trivial	$x_1+\dots+x_6=0$					signature

I SMF
II Constr° of vector valued (v.v.) SMF
III Proof of thm

I SMF

$\Gamma = \Gamma_2$ or $\Gamma_2[2]$

Déf: A SMF of weight $\rho: GL(2,\mathbb{C}) \to GL(W)$ fin. dim repr. is a holomorphic map $f: \mathcal{H}_2 \to W$ s.t.
$f(g.\tau) = \rho(c\tau+d) f(\tau)$ for all $\tau \in \mathcal{H}_2$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

If $\rho = \rho_1 \oplus \rho_2$ $M_\rho(\Gamma) \simeq M_{\rho_1}(\Gamma) \oplus M_{\rho_2}(\Gamma)$.
→ On ne considère que les repr° irréd de $GL(2,\mathbb{C}) \leftrightarrow (\lambda_1, \lambda_2) \in \mathbb{N}^2$ $\lambda_1 \geq \lambda_2$

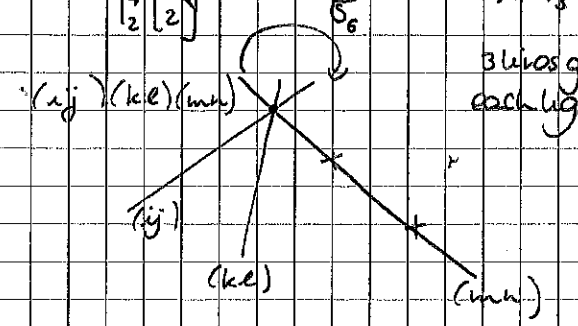
$Sym^j(V) \otimes \det^k(V) = \rho_{j,k}$ $(j+k, k) \rightsquigarrow M_{j,k}(\Gamma)$

Rq: j odd $\Rightarrow M_{j,k}(\Gamma) = \{0\}$ due to the fact $-I_4 \in \Gamma$.

From now on, j is even, $j>0$, then $M_{j,2k+1}(\Gamma) = S_{j,2k+1}(\Gamma)$

$M_{j,k}(\Gamma) \xrightarrow{\Phi} S_{j+k}(\Gamma_1)$

Satake compactification of $\Gamma_2[2] \backslash \mathcal{H}_2$



$\mathbb{P}^1$ lignes, pts
$(15, 15)_3$ configuration
3 lignes go th. a pt
each ligne has 3 pts

$M_{j,k}(\Gamma_2[2])$ is a $\mathfrak{S}_6$-repr. space

$R^{ev} = \bigoplus_{k \geq 0} M_{2k}(\Gamma_2[2]) \overset{\text{Igusa}}{=} \mathbb{C}[x_1, x_2, x_3, x_4, x_5] / (f)$ $\quad$ ($x_1 = \theta_1^4$, $x_2 = \theta_2^4$)

$\Gamma \to \begin{bmatrix} 00 \\ 00 \end{bmatrix}_{n_1}, \begin{bmatrix} 00 \\ 01 \end{bmatrix}_{n_2} \dots \begin{bmatrix} 11 \\ 11 \end{bmatrix}_{n_{10}}$

10 even, 6 odd

$\theta\begin{bmatrix} \mu \\ \nu \end{bmatrix}(\tau, z) = \sum_{n \in \mathbb{Z}^2} e^{i\pi (n + \frac{\mu}{2})(\tau (n+\frac{\mu}{2})^t + 2(z + \frac{\nu}{2})^t)}$ $\quad$ $\tau \in \mathcal{H}_2$, $z \in \mathbb{C}^2$

$\mu, \nu : [\mu_1 : \mu_2]$ $\mu_i \in \{0,1\}$

$z=0$, $\ z \equiv 0$

$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}_{n_4}$

$x_i = \theta_{n_i}^4(\tau, 0)$ → weight 2.

$x_6 = x_1 - x_2 - x_3 - x_4 + x_5$
$x_7, \dots, x_{10}$: autres relations linéaires

(f) $\mathbb{P}^4 \subset \mathbb{P}^9$ Igusa quartic $\left(\sum_{i=1}^{10} x_i^2 \right)^2 - 4 \sum_{i=1}^{10} x_i^4 = 0$

$Sym^2 \curvearrowleft M_2(\Gamma_2[2]) = s[2^3]$

$M_4(\Gamma_2[2]) = \underbrace{s[6]}_{= \text{triviale}} + s[4,2] + s[2^3]$

$M_6(\Gamma_2[2]) = \dots$

$M_8(\Gamma_2[2]) = \cancel{2}s[6] + \dots$ $\quad$ $\mathfrak{S}_6$ inv?

dim 70

$\chi_5 = \sqrt{\chi_{10}} = \prod_{i=1}^{10} \theta_i \in S_5(\Gamma_2[2]) \rightsquigarrow s[1^6]$

$M_k(\Gamma_2[2]) \times M_{j,k_1}(\Gamma_2[2]) \longrightarrow M_{j,k+k_1}(\Gamma_2[2])$

$(f,g) \longmapsto fg$

II Constructions

1 Rankin – Cohen brackets

$f,g : \mathcal{H}_2 \to \mathbb{C}$

$[f,g](\tau) = \frac{1}{2i\pi}\left(kf \frac{dg}{d\tau} - \ell g \frac{df}{d\tau}\right)(\tau)$

$\frac{d}{d\tau} = \begin{pmatrix} \frac{\partial}{\partial\tau_1} & \frac{1}{2} \frac{\partial}{\partial \tau_{12}} \\ \frac{1}{2}\frac{\partial}{\partial\tau_{12}} & \frac{\partial}{\partial\tau_2}\end{pmatrix}$ $\quad \begin{pmatrix}\tau_1 & \tau_{12}\\ \tau_{12} & \tau_2\end{pmatrix}$

Thm: $M_k(\Gamma) \times M_\ell(\Gamma) \longrightarrow M_{2,k+\ell}(\Gamma)$

$(f,g) \longmapsto [f,g] \in M_2(\mathbb{C})$

$\begin{bmatrix} a & b\\ c & d\end{bmatrix} \longrightarrow \begin{bmatrix} a\\ 2b\\ c\end{bmatrix}$ $\quad \mathrm{Sym}^2 V$

Ex: $F_{ij} = \underset{s[2^3]}{[x_i, x_j]} \in M_{2,4}(\Gamma_2[2]) \rightsquigarrow \Lambda^2(s[2^3]) = s[3,1^3]$

2 Gradient of odd theta

$\theta\begin{bmatrix}\mu\\ \nu\end{bmatrix}(\tau,z)$, $\quad z = \begin{pmatrix}z_1\\ z_2\end{pmatrix}$

$m_1,\dots,m_6 \rightsquigarrow G_i(\tau) = \begin{pmatrix}\frac{\partial\theta_{m_i}}{\partial z_1}\\ \frac{\partial\theta_{m_i}}{\partial z_2}\end{pmatrix}(\tau,0) \in M_{2,1/2}(\Gamma_2[4,8])$

$F = \theta_{i_1}\cdots\theta_{i_k}\,\mathrm{Sym}^j(G_{j_1},\dots,G_{i_j}) \in \underbrace{M_{j,\frac{k}{2}+\frac{j}{2}}(\Gamma_2[2])}$?

Thm (Igusa – SM): $M = \begin{bmatrix}\mu_1\\ \vdots\\ \mu_k\\ m_{i_1}\\ \vdots\\ m_{i_j}\end{bmatrix}$, $F \in$ $\quad$ if ${}^tMM \equiv 0\ [4]$

$[\mu_1,\mu_2,\nu_1,\nu_2]$

Ex: $G_{12} = \theta_1\theta_2\cdots\theta_6\,\mathrm{Sym}^2(G_1,G_2) \in M_{2,4}(\Gamma_2[2])$

$m_1, m_2 \longrightarrow \underset{\text{even}}{m_k = m_1+m_2} + a = b+c+d$

s.t. $\{m_1,m_2,a,b,c,d\} = \{m_1,\dots,m_6\}$

$15\ G_{ij} \rightsquigarrow s[3,1^3] \oplus s[2,4]$

$M_{2,4}(\Gamma_2[2])$

$F_{12} = \frac{1}{\varrho^2}\left(-G_{12} + G_{56} - G_{6} - G_{26}\right)$.

III Proof $\quad Gp = \Gamma_2[2]$

Strategy: B., F., & vdG suggest same decomposition of $M_{2,4}, M_{2,6}, M_{2,8}, M_{2,10}$

Conjecture(1): $M_{2,4}(\Gamma_2[2]) = s[3,1^3] + s[2,1^4]$ ★

$\rightsquigarrow$ suggests 15 generators; we already got them.

They also suggest relations:

Conjecture(2): $M_{2,6}(\Gamma_2[2]) = s[4,2] + 2s[3,2,1] + s[3,1^3] + s[2^3]$

Tensor ★ by $\underset{\substack{\cong\\ s[2^3]}}{M_2(\Gamma_2[2])} \subseteq M_{2,6}(\Gamma_2[2])$

gives: $s[4,2] + \boxed{s[4,1^2]} + s[3,2,1] + s[3,1^3] + s[2^3]$

$+ s[4,2] + s[3,2,1]$

on top: donne une relation

$[x_j, x_k]$

$x_i F_{jk} - x_j F_{k,i} + x_k F_{ij} = 0$

$M_4(\Gamma_2[2]) \ominus s[4,2] + s[4,1^2] \quad \otimes M_2(\Gamma_2[2])$

$\vdots$

$\underbrace{x_1 R_{234} - x_2 R_{134} \cdots\cdots}_{=} = 0$

? 61b construction se répète avec périodicité 8, pas toujours

Dijke

$T = \mathbb{C}[x_1,\dots,x_6] \quad R^{ev} = T - T(-4) \quad \mathcal{M}_2$

Castelnuovo - Mumford regularity: The vector bundle $Sym^2(E)$ is 3-regular for $\det(E)^2$

Comes from the vanishing of

$X = \tilde{\mathcal{A}}_2[\Gamma_2[2]]$

$H^1(X, Sym^2 \otimes \det^4)$
$H^2(X, Sym^2 \otimes \det^2)$ } quite hard

$$H^3(X, Sym^2) \underset{\text{Serre}}{\simeq} H^0(X, Sym^2(E) \otimes \mathcal{O}(\underset{\text{diviseur}}{D}))$$

$$\wr\!\wr \quad S_{2,0}(\Gamma_2[2]) = 0 ?$$

Assume that $S_{2,0}(\Gamma_2[2]) \neq 0$

$$\left(m_{s[6]}\, s[6] \oplus \cdots \oplus m_{s[1^6]}\, s[1^6] \right) \star \otimes \chi_5 \Rightarrow s[4,2]$$

$$S_{2,5}(\Gamma_2[2]) = s[2^2, 1^2] \Rightarrow \star \text{ il n'y a que,}$$

puis on tensorise par autre chose par nature qu'il n'apparaît pas.