Clap-Clap : Formes modulaires, aspects théoriques et calculatoires
Rennes, lundi 6 février 2017, 11h
Francesco Lemma

Siegel modular forms of genus 2 I

Réf: I. Skoruppa Computations of Siegel m. f. of gen. 2. Math Comp 98

II. Asgari-Schmidt Siegel modular forms and representations

III. Tilouine Cohomologie des variétés de Siegel, Publi. Math. Besançon

I Classical pt of view

II Automorphic point of view

III Galois pt of view

I Classical point of view

$\psi = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$ symplectic form $/\mathbb{Z}^4$

$GSp(4) = \{ g \in GL(4) \,/\, {}^t g \psi g = \nu(g)\psi, \nu(g) \in \mathbb{G}_m \}$ reductive gp $(/\mathbb{Z})$

$Sp(4) = \mathrm{Ker}\, \nu$

Siegel half plane: $\mathcal{H}_2 = \{ Z \in M_2(\mathbb{C}) \,/\, {}^t Z = Z, \mathrm{Im}\, Z \text{ positive definite} \} \underset{\text{open}}{\subseteq} \mathbb{C}^3$

$GSp(4, \mathbb{R})_+ = \nu^{-1}(\mathbb{R}_+^*) \circlearrowright \mathcal{H}_2$ via $\begin{pmatrix} A & B \\ C & D \end{pmatrix} . Z = (AZ + B)(ZC + D)^{-1}$

Facts: (1) the action is transitive

(2) $\mathrm{Stab}(i I_2) = \mathbb{R}_+^* U(2)$

where $U(2) = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in Sp(4, \mathbb{R}) \,/\, A^t A + B^t B = I_2 \right\}$

max. cpct subgp of $Sp(4, \mathbb{R})$

$\mathcal{H}_2 \simeq GSp(4, \mathbb{R}) / \mathbb{R}_+^* U(2) \simeq Sp(4, \mathbb{R}) / U(2)$

Let $(k_1, k_2) \in \mathbb{Z}^2$, $k_1 \geq k_2$ $V_{k_1, k_2} = \mathrm{Sym}^{k_1 - k_2}(\mathbb{C}^2) \otimes \det_{GL_2(\mathbb{C})}^{k_2}$

irred. alg. repr

$\rho_{k_1, k_2} : GL(2, \mathbb{C}) \to GL(V_{k_1, k_2})$

of highest weight (k_1, k_2).

$\Gamma := Sp(4, \mathbb{Z})$ = niveau 1

Déf: A Siegel modular form (of genus 2) of weight (k_1, k_2) and level Γ is a holomorphic function $f : \mathcal{H}_2 \to V_{k_1, k_2}$ s.t.

$$f(\gamma Z) = \rho_{k_1, k_2}(CZ + D) f(Z) \quad \text{where } \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma, \; Z \in \mathcal{H}_2$$

Notation: $M_{k_1, k_2}(\Gamma)$; when $k_1 = k_2 = k$ $M_k(\Gamma)$

Consider $k_1 = k_2 = k$, $f \in M_k(\Gamma)$

$\forall S \in M(2, \mathbb{Z})$ tq ${}^t S = S$, $\begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} \in \Gamma$ $f(Z + S) = f(Z)$ $\forall Z \in \mathcal{H}_2$

Fourier expansion $f(Z) = \sum_{n, r, m \in \mathbb{Z}} a(n, r, m) e^{2i\pi n\tau} e^{2i\pi r z} e^{2i\pi m \tau'}$

where $Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}$ $\mathrm{Im}\,\tau, \mathrm{Im}\,\tau' > 0$ $(\mathrm{Im}\, z)^2 - \mathrm{Im}\,\tau\, \mathrm{Im}\,\tau' < 0$.

Prop (Koecher principle): $a(n, r, m) = 0$ if $f\begin{pmatrix} n & \frac{r}{2} \\ \frac{r}{2} & m \end{pmatrix}$ is not positive.

$$\text{So } f(Z) = \sum_{\substack{r^2 - 4mn \leq 0 \\ m, n \geq 0 \\ r \in \mathbb{Z}}} a(n, r, m) q^n \zeta^r q'^m$$

$$= \sum_{\substack{N = \begin{pmatrix} n & \frac{r}{2} \\ \frac{r}{2} & m \end{pmatrix} \\ \text{half integral} \\ \text{positive}}} a(N) e^{2i\pi \mathrm{Tr}(NZ)}$$

Hecke operators $T(p) = \Gamma \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & p & \\ & & & p \end{pmatrix} \Gamma$

$$T(p^2) = \Gamma \begin{pmatrix} 1 & & & \\ & p & & \\ & & p^2 & \\ & & & p \end{pmatrix} \Gamma$$

$$\Gamma h \Gamma = \coprod_i \Gamma h_i \qquad h_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} \in GSp(4,\mathbb{R})_+$$

$$Tf = \sum_i f|_k h_i \qquad \text{where } \left(f|_k h_i\right)(Z) = \nu(h)^k \det(C_i Z + D_i)^k f(h_i Z)$$

$$a_{T(p)f}(n,r,m) = a_f(pn, pr, pm) + p^{2k-3} a_f\left(\frac{n}{p}, \frac{r}{p}, \frac{m}{p}\right) + p^{k-2} a_f\left(\frac{m}{p}, r, pn\right) + p^{k-2} \sum_{\nu=0}^{p-1} a_f\left(\frac{n+r\nu+m\nu^2}{p}, r+2m\nu, pm\right)$$

where $a(\varphi) = 0$ if φ not integral

Let f be an eigenform for all $T(p)$ and $T(p^2)$.

$\forall$ p prime $\quad T(p)(f) = \lambda_p f$

$\qquad T(p^2) f = \lambda_{p^2} f$

Degree four L-function:

$$L(f,s) = \prod_p Q_p(p^{-s})^{-1} \qquad \text{for } \operatorname{Re} s \gg 0$$

$$Q_p(X) = 1 - \lambda_p X + (\lambda_{p^2} - \lambda_p - p^{2k-4}) X^2 - \lambda_p p^{2k-3} X^3 + p^{4k-6} X^4$$

For $f \in M_k(\Gamma)$, Fourier-Jacobi extension:

$$f(Z) = \sum a(n,r,m) q^n \zeta^r q'^m = \sum \phi_m(\tau, z) q'^m$$

Poincaré half plane

$\phi_m : \mathcal{H}_1 \times \mathbb{C} \to \mathbb{C} \qquad \phi_m \in J_{k,m}$ Jacobi forms of weight k index

(Eichler - Zagier)

$$J_{k,0} = M_k(SL(2,\mathbb{Z}))$$

Def: f is cuspidal if $\phi_0 = \sum_{n \geq 0} a(n,0,0) q^n$

$$= \lim_{t \to \infty} f\begin{pmatrix} \tau & 0 \\ 0 & it \end{pmatrix} \text{ vanishes}$$

Notation: $S_k(\Gamma) \subseteq M_k(\Gamma)$ space of cusp forms.

We have

$$\begin{tikzcd} M_k(\Gamma) \arrow[r, "\phi_0", "K"'] & M_k(SL(2,\mathbb{Z})) \arrow[d, phantom, "\cup" description] \\ & S_k(SL(2,\mathbb{Z})) \ni g \arrow[ul, bend left] \end{tikzcd}$$

$K =$ Klingen - Eisenstein series

$$\hat{g}(Z) = g(\tau) \qquad Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}$$

$$(Kg)(Z) = \sum_{\gamma \in P \backslash \Gamma} \left(\hat{g}|_k \gamma\right)(Z) \qquad P = \begin{pmatrix} a & * & b & * \\ * & \pm 1 & * & * \\ c & & d & * \\ & & & \pm 1 \end{pmatrix} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

converges for $k > 4$ even

Prop: $K : S_k(SL(2,\mathbb{Z})) \to M_k(\Gamma)$ is such that $\phi_0 Kg = g$

Maaß Spezialschar

$\phi \in J_{k,1} \qquad \phi : \mathcal{H} \times \mathbb{C} \to \mathbb{C} \qquad$ Fourier $\displaystyle \phi(\tau, z) = \sum_{\substack{r \in \mathbb{Z} \\ D \leq 0 \\ r^2 \equiv D \bmod 4}} c(D) q^{\frac{r^2-D}{4}} \zeta^r$

Thm (Maaß): The map $J_{k,1} \to$

$$\phi \mapsto \sum a(n,r,m) q^n \zeta^r q'^m$$

where $a(n,r,m) = \begin{cases} \displaystyle\sum_{a|(n,r,m)} a^{k-1} c\left(\frac{r^2-4mn}{a^2}\right) & \text{if } (n,r,m) \neq 0 \\ -\dfrac{B_{2k}}{4k} c(0) & \text{if } (n,r,m) = 0 \end{cases}$

defines an embedding $V : J_{k,1} \to M_k(\Gamma)$

s.t. the first F-J. coeff of $V\phi$ is ϕ.

and mapping cusp forms to cusp forms.

$VJ_{k,1}$ is called Maaß Spezialschar.

Prop (Skoruppa): The map $(f,g) \longrightarrow \frac{k}{2} Af + \left(q\frac{d}{dq} f\right)B + gB$

defines an isomorphism

$$I : M_k(SL(2,\mathbb{Z})) \oplus S_{k+2}(SL(2,\mathbb{Z})) \simeq J_{k,1}$$

where A and B are some explicit functions on $\mathcal{H}_1 \times \mathbb{C}$.

and $I(f,g)$ is cuspidal iff f is cuspidal.

$$\Delta := q\prod_{n=1}^{\infty}(1-q^n)^{24} \in S_{12}(SL(2,\mathbb{Z})) \qquad E_{2k} = 1 - \frac{4k}{B_{2k}}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^n$$

$$\in M_{2k}(SL(2,\mathbb{Z}))$$

Let $\psi_4 = VI(E_4, 0)$ $\quad \psi_6 = VI(E_6, 0)$ not cuspidal

$\mathcal{X}_{10} = VI(0, -\Delta)$ $\quad \mathcal{X}_{12} = VI(\Delta, 0)$ cuspidal

Thm (Igusa): $\bigoplus_{k\in\mathbb{Z}} M_{2k}(\Gamma) = \mathbb{C}[\psi_4, \psi_6, \mathcal{X}_{10}, \mathcal{X}_{12}]$ Pas de relation ! (si on s'en tient aux poids pairs)

Rmq: * there is a thm for all weight

* the first modular form of odd weight shows up in weight $\mathcal{X}_{35}$.

$S_{k,1} \subsetneq J_{k,1}$ Jacobi cusps forms $\qquad VS_{k,1} \subsetneq S_k(\Gamma)$

Thm (Saito - Kurokawa conjecture, Maaß, Andrianov)

$VS_{k,1}$ is generated by eigenforms and those correspond to eigenforms in $S_{2k-2}(SL(2,\mathbb{Z}))$ in such a way that if $F \leftrightarrow f$, we have

$$L(F,s) = \zeta(s-k+1)\,\zeta(s-k+2)\,L(f,1)$$

$\forall F \in VS_{k,1}$ eigenform $\underset{\text{Oda}}{\overset{\Rightarrow}{\Leftarrow}}$ $L(F,s)$ has poles

$S_k^h(\Gamma) :=$ subspace of $S_k(\Gamma)$ generated by eigenforms F s.t.

$L(F,s)$ is holomorphic on $\mathbb{C}$.

$$M_k(\Gamma) = \underset{\text{Klingen Eisenstein series}}{KS_k(SL(2,\mathbb{Z}))} \oplus \underset{\text{Special}}{VJ_{k,1}} \oplus S_k^h(\Gamma)$$ Hecke invariant decompos?

$$\sum_{k=0}^{\infty} \dim S_{2k}^h(\Gamma)\, X^{2k} = \frac{X^{20}(1+X^2+X^4-X^{12}-X^{14})}{(1-X^4)(1-X^6)(1-X^{10})(1-X^{12})}$$

In particular, $\min\{k \,/\, S_k^h(\Gamma) \neq \{0\}\} = 20$ and $\dim S_{20}^h(\Gamma) = 1$.

Thm (Skoruppa): $S_{20}^h(\Gamma)$ is generated by $\Upsilon_{20} = -2^9 3^2 5\, 7 . 11 . \mathcal{X}_{10}^2$

$$+ V\left(\frac{1}{2}\phi_{12}E_4^2 + \frac{1}{2}\phi_{10}E_4E_6\right)$$

and all the Fourier coeff of Υ_{20} are in $\mathbb{Z}$.

"$\underline{D}$" $\quad S_{20}(\Gamma) = VS_{20,1} \oplus S_{20}^h(\Gamma)$

$\mathcal{X}_{10}^2 \notin VS_{20,1}$ because its first Fourier Jacobi coeff vanishes

Up to normalisation $\Upsilon_{20} = \mathcal{X}_{10}^2 + aV_1 + bV_2$ $\qquad V_1 = V\left(\frac{1}{2}\phi_{12}E_4^2\right)$

$$B = \begin{pmatrix} \mathcal{X}_{10}^2 \\ V_1 \\ V_2 \end{pmatrix} \qquad T(2)B = MB \qquad \underset{\text{compute}}{M \in M(3,\mathbb{C})} \qquad V_2 = V\left(\frac{1}{2}\phi_{10}E_4E_6\right)$$

Let $\mathcal{X}$ be the charact. polynomial of $T(2)|_{VS_{20,1}}$

${}^tB\,\mathcal{X}(M) = (*\Upsilon_{20}, 0, 0)$

$$L_2(F,s) = \zeta_2(s-k+1)\,\zeta_2(s-k+2)\,L_2(f,s)$$

$\Rightarrow \mathcal{X}(X) = \tilde{\mathcal{X}}(X + 2^{k-1} + 2^{k-2})$

$\tilde{\mathcal{X}}$ char. poly. of $T(2)$ on $S_{38}(SL(2,\mathbb{Z}))$

□

II Automorphic point of view

$G = GSp(4)$ $\quad \mathbb{A} = \mathbb{R} \times \mathbb{A}_f$ adeles of $\mathbb{Q}$, $\mathbb{Q} \hookrightarrow \mathbb{A}$ diagonally.

Strong approximation thm: $G(\mathbb{A}) = G(\mathbb{Q}) G(\mathbb{R})_+ G(\hat{\mathbb{Z}})$

let $f \in M_k(\Gamma)$, define $\phi_f : G(\mathbb{A}) \longrightarrow \mathbb{C}$

$$g = g_\mathbb{Q} g_\infty \ell \longmapsto \underset{\text{facteur de similitude}}{\nu(g_\infty)^k} \det(C i I_2 + D)^{-k} f(g_\infty i I_2)$$

$$g_\infty = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Well defined: $g_\mathbb{Q} g_\infty \ell = g'_\mathbb{Q} g'_\infty \ell'$

$$\Rightarrow g'^{-1}_\infty g_\infty \ell'^{-1} \ell \in \underbrace{G(\mathbb{Q}) \cap (G(\mathbb{R})_+ G(\hat{\mathbb{Z}}))}_{\Gamma = Sp(4,\mathbb{Z})}$$

+ f modular.

Properties
(1) $\phi_f(pg) = \phi_f(g)$ $\quad \forall p \in G(\mathbb{Q})$ $\forall g \in G(\mathbb{A})$
(2) $\phi_f(g\ell) = \phi_f(g)$ $\quad \forall \ell \in G(\hat{\mathbb{Z}})$
(3) $\phi_f(gz) = \phi_f(g)$ $\quad \forall z \in Z(\mathbb{A})$ $\quad Z \subseteq G$ centre.

$$\leadsto \phi_f : Z(\mathbb{A}) G(\mathbb{Q}) \backslash G(\mathbb{A}) \longrightarrow \mathbb{C}$$

(4) $\mathfrak{p}^+$ = holomorphic tangent space at iI_2 to $\mathcal{H}_2$
$\mathfrak{p}^-$ = antiholomorphic ———
via $\mathcal{H}_2 \simeq Sp(4,\mathbb{R}) / U(2)$, we have $\mathfrak{p}^\pm = \left\{ \begin{pmatrix} A & \pm iA \\ \pm iA & A \end{pmatrix} \in \mathfrak{g},\ {}^tA = A \right\}$

$$\mathfrak{g} = (\mathrm{Lie}\, G)_\mathbb{C}$$

f holomorphic $\Leftrightarrow$ $\mathfrak{p}^- \phi_f = 0$ where $X \in \mathfrak{g}$ acts on ϕ_f by

$$(X . \phi_f)(g) = \frac{d}{dt}_{t=0} \phi_f(g \exp(tX))$$

Prop:

f is cuspidal $\Leftrightarrow$ $\forall P = MN$ proper parabolic subgp of G
N unipotent radical
M Levi subgp

we have $\displaystyle\int_{N(\mathbb{Q}) \backslash N(\mathbb{A})} \phi_f(ng)\, dn = 0$

Rq: up to conjugacy, there are 3 proper parabolic in G

Borel: $G \cap \{\text{upper triangular matrices}\}$

$Q = \mathrm{Stab}_G \langle e_1, e_2 \rangle$

isotropic subspaces

$P = \mathrm{Stab}_G \langle e_1 \rangle$ $\qquad$ where $\mathbb{Z}^4 = \langle e_1, e_2, e_3, e_4 \rangle$

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$\boxed{\text{Siegel modular forms of genus 2' II}}$

Petersson inner product

$\mathcal{H}_2$ = Siegel upper half plane

$$Z \in \mathcal{H}_2, \quad Z = \begin{pmatrix} x_{11}+iy_{11} & x_{12}+iy_{12} \\ \cdots & \cdots \end{pmatrix} \qquad dZ = (\det \operatorname{Im} Z)^{-3} \prod_{i \leq j} dx_{ij}\, dy_{ij}$$

volume form on $\mathcal{H}_2$ invariant by Γ

Let $f_1, f_2 \in M_k(\Gamma)$ s.t. f_1 or f_2 is in $S_k(\Gamma)$.

Def: $$\langle f_1, f_2 \rangle = \int_F \det(\operatorname{Im} Z)^k f_1(Z)\overline{f_2(Z)}\, dZ \quad \text{convergent}$$

F fundamental domain of Γ on $\mathcal{H}_2$

II Adelic point of view

$G = GSp(4) \xrightarrow{\nu} \mathbb{G}_m \qquad \mathbb{A} = \mathbb{R} \times \mathbb{A}_f$ adeles of $\mathbb{Q}$.

$$\begin{tikzcd} f: i I_2 \in \mathcal{H}_2 \arrow[r] & \mathbb{C} \\ I_4 \arrow[u] & G(\mathbb{R})_+ / \mathbb{R}_+^* U(2) \arrow[ul, phantom, "\cong"] \end{tikzcd}$$

Strong approximation thm

$$G(\mathbb{A}) = G(\mathbb{Q})\, G(\mathbb{R})_+\, G(\hat{\mathbb{Z}})$$

$$\phi_f(g) = \nu^k(g_\infty) \det(C i I_2 + D)^{-k} f(g_\infty\, i I_2)$$

$$g \in G(\mathbb{A}) \quad g = g_{\mathbb{Q}}\, g_\infty\, \ell \qquad g_\infty = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Lemma. If $f \in M_k(\Gamma)$, ϕ_f is well defined

In fact, $\phi_f : Z(\mathbb{A})G(\mathbb{Q}) \backslash G(\mathbb{A}) \longrightarrow \mathbb{C}$ where $Z \subseteq G$ is the center.

We are interested in $$L^2\left(Z(\mathbb{A})G(\mathbb{Q}) \backslash G(\mathbb{A})\right) = \left\{ \varphi : Z(\mathbb{A})G(\mathbb{Q}) \backslash G(\mathbb{A}) \to \mathbb{C}, \int_{Z(\mathbb{A})G(\mathbb{Q}) \backslash G(\mathbb{A})} |\varphi(g)|^2 dg < \infty \right\} \Big/ \sim$$

$G(\mathbb{A}) \curvearrowright \quad \cup\!|$

action by right translation $\qquad L^2_0$ cusps forms

Lemma: We have an isomorphism $\Gamma\backslash\mathcal{H}_2 \xrightarrow{\sim} Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})/U(2)G(\hat{\mathbb{Z}})$

$$g_\infty \cdot iI_2 \longleftarrow\!\shortmid \ \overline{g = g_{\mathbb{Q}}\, g_\infty\, l}$$

s.t. $d\bar{z}$ corresponds to a scalar multiple of dg.

Corollary: $f \in S_k(\Gamma) \Rightarrow \phi_f \in L_0^2$

We know: $$\underset{\substack{\circlearrowleft\\ G(\mathbb{A})}}{L_0^2} = \underset{\pi \text{ irreducible automorphic cuspidal representation of } G(\mathbb{A})}{\bigoplus} m(\pi)\,\pi \qquad m(\pi)\in\mathbb{N} \text{ finite multiplicity}$$

invariant by $G(\mathbb{Q})$

let $V_f \subseteq L_0^2$ be the space generated under right translation of $G(\mathbb{A})$ by ϕ_f and let $\pi \subseteq V_f$ be an irreducible component.

$$\pi = \underset{\nu \text{ places of } \mathbb{Q}}{\bigotimes}{}' \pi_\nu .$$

We would like to describe π_p $\forall p<\infty$ and π_∞.

$f \in S_k(\underline{\underline{\Gamma}}) \Rightarrow \phi_f$ invariant by right translation under $G(\hat{\mathbb{Z}})$

$\Rightarrow \forall p<\infty \quad \pi_p^{G(\mathbb{Z}_p)} \neq \{0\}$ (unramified representation)

$B = \underset{\substack{\Downarrow\\ T}}{T}N$ Borel subgp $\quad T = \left\{\begin{pmatrix} x_1 &&& 0\\ & x_2 && \\ && x_1^{-1}\nu & \\ 0 &&& x_2^{-1}\nu \end{pmatrix}\right\} \subseteq G$

let $\chi_0, \chi_1, \chi_2 : \mathbb{Q}_p^\times \longrightarrow \mathbb{C}^\times$ s.t. $\chi_i|_{\mathbb{Z}_p^\times} = 1$ continuous characters

$$\chi_0\times\chi_1\times\chi_2 \quad T(\mathbb{Q}_p) \longrightarrow \mathbb{C}^\times$$

$$\begin{pmatrix} x_1 &&& 0\\ & x_2 && \\ && x_1^{-1}\nu & \\ 0 &&& x_2^{-1}\nu \end{pmatrix} \mapsto \chi_1(x_1)\chi_2(x_2)\chi_0(\nu)$$

and extend to $\chi_0\times\chi_1\times\chi_2 : B(\mathbb{Q}_p) \longrightarrow \mathbb{C}^\times$ using proj° $B(\mathbb{Q}_p) \twoheadrightarrow T(\mathbb{Q}_p)$.

let $$\underset{\substack{\circlearrowleft\\ G(\mathbb{Q}_p)}}{\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}}(\chi_0\times\chi_1\times\chi_2) = \left\{ f : G(\mathbb{Q}_p) \to \mathbb{C} \;/\; \begin{array}{l} \forall b \in B(\mathbb{Q}_p) \\ f(bg) = (\chi_0\times\chi_1\times\chi_2)(b) f(g) \end{array} \right\}$$

Lemma: $\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}(\chi_0\times\chi_1\times\chi_2)$ is unramified.

D) Iwasawa decomp° $G(\mathbb{Q}_p) = B(\mathbb{Q}_p)G(\mathbb{Z}_p)$

$\Rightarrow \varphi \in \mathrm{Ind}(\dots)$ uniquely determined by $\varphi|_{G(\mathbb{Z}_p)}$

let $\varphi_0 = 1$ on $G(\mathbb{Z}_p)$. It is invariant by $G(\mathbb{Z}_p)$ $\square$

Thm: $\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}(\chi_0\times\chi_1\times\chi_2)$ has a unique unramified irreducible subquotient and every π_p is of this form.

↳ unramified, pas forcément venant de formes modulaires.

Satake parameters: $b_i = \chi_i(p) \in \mathbb{C}^\times$

$$L(s,\pi_p,\mathrm{spin})^{-1} = (1-b_0p^{-s})(1-b_0b_1p^{-s})(1-b_0b_2p^{-s})(1-b_0b_1b_2p^{-s})$$

Prop: Assume f is an eigenform for $T(p)$ & $T(p^2)$.

Then $L(s,\pi_p,\mathrm{spin}) = L_p\left(s-\frac{1}{2}+k, f\right)$

defined yesterday.

We want to describe π_∞:

Interlude

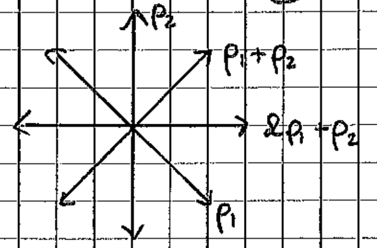
~ on irreducible representations of G (algebraic) ~

$$X^*(T) = \mathrm{Hom}(T,\mathbb{G}_m) \xleftarrow{\sim} \{(k,k',c) \;/\; k+k' \equiv c \bmod 2\}$$

$$\begin{pmatrix} x_1 &&& 0\\ & x_2 && \\ && x_1^{-1}\nu & \\ 0 &&& x_2^{-1}\nu \end{pmatrix} \mapsto x_1^{k}x_2^{k'}\nu^{\frac{c-k-k'}{2}} \longleftarrow\!\shortmid (k,k',c) \quad \text{“weights”}$$

let $\rho_1 = (1,-1,0)$ $\rho_2 = (0,1,0)$

The roots, which are the weights appearing in the adjoint action of T on Lie G are:



for $B = TN$, T as before and $N = \left\{ \underbrace{\begin{pmatrix} 1 & x_0 & & 0 \\ & 1 & & \\ 0 & & 1 & \\ & & -x_0 & 1 \end{pmatrix} \begin{pmatrix} 1 & & x_1 & x_2 \\ & 1 & x_3 & x_4 \\ & & 1 & \\ & & & 1 \end{pmatrix}}_{\text{unipotent radical}}, x_i \in \mathbb{G}_a \right\}$

The positive roots which are the roots showing up in Lie B are 

Thm: $\forall (k,k',c) \in \mathbb{Z}^3$, $h+h' \equiv c \bmod 2$, $k \geq k' \geq 0$, there is, up to equivalence, a unique algebraic irreducible representation of G in $V_{k,k',c}$ finite dim^a.

$(\mathfrak{g}, K_\infty)$-cohomology: $\mathfrak{g} = (\mathrm{Lie}\, G)_{\mathbb{C}}$ $K_\infty = \mathbb{R}_+^* U(2) \subseteq G(\mathbb{R})_+$

Borel-Wallach: let V be a $(\mathfrak{g}, K_\infty)$-module (can think repr° of $G(\mathbb{R})$ in a Hilbert space)

Construct a complex $C^*(\mathfrak{g}, K_\infty, V) = \mathrm{Hom}_{K_\infty}(\Lambda^* \mathfrak{g}/K, V)$

$K = (\mathrm{Lie}\, K_\infty)_{\mathbb{C}}$.

Def: A $(\mathfrak{g}, K_\infty)$-module π_∞ is cohomological, in degree 3, for weight (k,h',c)

if $H^3(\mathfrak{g}, K_\infty, \pi_\infty \otimes V_{k,k',c}) \neq 0$

Thm (Harish-Chandra, Vogan-Zuckerman, Borel-Wallach)

$\forall (k,k',c)$ $k \geq k' \geq 0$ $k+k' \equiv c \bmod 2$

up to equivalence, there are four $(\mathfrak{g}, K_\infty)$-modules $\pi_\infty^H, \pi_\infty^W, \bar{\pi}_\infty^H, \bar{\pi}_\infty^W$ cohom^d, in deg 3, of weight (k,k',c),

which are characterized by $H^3(\mathfrak{g}, K_\infty, \pi_\infty^H \otimes V_{k,k',c}) = \mathrm{Hom}_{K_\infty}(\Lambda^3 \mathfrak{g}/K, \pi_\infty^H \otimes V_{k,k',c})$

$= \mathrm{Hom}(\Lambda^3 \mathfrak{p}^+, \pi_\infty^H \otimes V_{k,k',c})$

and this space is of dim 1.

+ relations similaires pour les 3 autres

Prop. If $k \geq 3$, $f \in S_k(\Gamma)$, then the archimedian component of π attached to f is isomorphic to π_∞^H for weight $(k-3, k-3, 2k-6)$.

More generally $f \in S_{k_1,k_2}(\Gamma)$ $\longleftrightarrow$ π_∞^H of weight
$k_2 \geq 3$ $(k_1-3, k_2-3, k_1+k_2-6)$

Rk: For $k=2$, the archimedian repr° is not cohomological.

III Galois point of view

let $(k,k') \in \mathbb{Z}$, $k \geq k' \geq 0$. let $\pi = \overset{\text{archimedian}}{\pi_\infty} \otimes \overset{\text{non archimedian}}{\pi_f}$

a cuspidal automorphic representation of $G(\mathbb{A})$ such that

π_∞ is cohomological for weight $(k,k',k+k')$.

let N be the product of ℓ primes s.t. $\pi_\ell^{G(\mathbb{Z}_\ell)} = \{0\}$.

let $\bar{\mathbb{Q}} \subseteq \bar{\mathbb{Q}}_p$

Thm (Taylor, Laumon, Weissauer):

$\exists \rho_\pi : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GL}_4(\bar{\mathbb{Q}}_p)$ which is continuous and unramified outside Np and s.t. $\forall \ell \nmid Np$,

$\det(1 - \mathrm{Frob}_\ell X, \rho_\pi) = Q_\ell(X)$

where $Q_\ell(\ell^{-s}) = L(s, \pi_\ell, \mathrm{spin})^{-1}$

Def: π is cuspidal associated to parabolic (CAP) if $\exists$ proper parabolic subgp $P = \overset{\text{Levi}}{M}\overset{\text{unipotent}}{N}$, $M \twoheadrightarrow M$

$\exists$ cuspidal autom. repr. σ of $M(\mathbb{A})$ s.t.

if $\Pi = \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \sigma$ then $\pi_v \simeq \Pi_v$ at almost all places v.

Ex: this corresponds to $VS_{k,1}$ Maaß Spezialschar

Thm (Weissauer): let π be as above and assume π is not CAP.
Then the roots of Q_ℓ are algebraic of absolute value $p^{-w/2}$, where $w = k+k'+3$

Note: For $f \in VS_{k,1}$, $L(f,s) = \zeta(s-k+1)\zeta(s-k+2) \times$ degree 2

$$Q_p(X) = (1 - p^{k-1}X)(1 - p^{k-2}X) \ldots$$

so the thm is false (motif non pur, mais somme de motifs de poids différents)

Def. π is weakly endoscopic if $\exists\, \pi_1, \pi_2$ cusp. repr. of $GL(2, \mathbb{A})$ s.t. at almost every place v we have $L(s, \pi_v, \mathrm{spin}) = L(s, \pi_{1,v}) L(s, \pi_{2,v})$

Thm (Weissauer): If π is neither CAP nor weakly endoscopic, then $\rho_{\pi|D_p}$ is Hodge-Tate with Hodge-Tate weights: $0, -k'-1, -k-2, -k-k'-3$.

Furthermore, if $p \nmid N$, $\rho_{\pi|D_p}$ (Faltings)

Thm (Urban) $\det(1 - \phi X, D_{cris}(\rho_\pi)) = Q_p(X)$

Cohomology of Siegel varieties of dim 3.

$$\underset{\substack{\circlearrowleft \\ G(\mathbb{R})}}{\mathcal{H}_2^{\pm}} = \mathcal{H}_2 \amalg -\mathcal{H}_2 \subseteq \mathbb{C}^3 \qquad K \subseteq G(\mathbb{A}_f) \text{ cpct open subgp}$$

let $S_K^{an} = G(\mathbb{Q}) \backslash \mathcal{H}^{\pm} \times G(\mathbb{A}_f) / K$

If $K = K(N) = \mathrm{Ker}\left(G(\hat{\mathbb{Z}}) \longrightarrow G(\mathbb{Z}/N\mathbb{Z})\right) \quad N \geq 3$

Then $S_{K(N)}^{an} = \coprod_{(\mathbb{Z}/N\mathbb{Z})^{\times}} \Gamma(N) \backslash \mathcal{H}_2$ ✓ where $\Gamma(N) = \mathrm{Ker}(Sp(4,\mathbb{Z}) \to Sp(4, \mathbb{Z}/N\mathbb{Z}))$

$$\Gamma(N)\backslash\mathcal{H}_2 \simeq \{(A, \lambda, \eta)\}/\sim \qquad \text{where}$$

$$Z \longmapsto \mathbb{C}^g / (\mathbb{Z}^g + Z\mathbb{Z}^g)$$

polarisation donnée par $\mathrm{Im}\, Z$

$A/\mathbb{C}$ ab. surface

λ ppal pol° $A \simeq A^{\vee}$

η full level N structure $(\mathbb{Z}/N\mathbb{Z})^{2g} \simeq A[N]$

S_K^{an} has a canonical model $/\mathbb{Q}$ which is a quasi-proj smooth (if K small enough) variety.

let $L \subseteq G(\mathbb{A}_f)$ s.t. $gLg^{-1} \subseteq K \qquad S_L \xrightarrow{[\cdot g]} S_K$

gives action of $G(\mathbb{A}_f)$ on $(S_K)_{K \subseteq G(\mathbb{A}_f)}$

let $V_{k,k',c}$ irred repr of $G \rightsquigarrow$ local system on S_K, $\forall K$.

Ex: $V_{1,0,1}$ = standard repr. of $G \rightsquigarrow R^1\pi_* \mathbb{Q}_{(\ell)}$ where $\begin{matrix} A \\ \pi\downarrow \\ S_K \end{matrix}$ is the universal object.

$$H^3_!(S,V) = \varinjlim_K H^3_{dR}(S_K, V) = \varinjlim_K \left(H^3_c(S_K,V) \longrightarrow H^3(S_K,V)\right)$$

$$= H^3\left(\mathfrak{g}, K_\infty, \mathcal{C}^{\infty,(2)}\left(Z(\mathbb{A})G(\mathbb{Q}) \backslash G(\mathbb{A})\right) \otimes V\right)$$

$$\underset{\substack{\circlearrowleft \\ G(\mathbb{A}_f)}}{H^3_!(S,V)} \simeq \bigoplus_{\substack{\pi \text{ cusp.} \\ \text{aut. irred.} \\ \text{repr of } G(\mathbb{A})}} \underbrace{H^3(\mathfrak{g}, K_\infty, \pi_\infty \otimes V)}_{\dim 1}{}^{m(\pi)} \otimes \pi_f$$

let $W_{\pi_f} = \mathrm{Hom}(\pi_f, H^3_!(S,V)) \quad \dim\ m(\pi_\infty^H \otimes \pi_f) + m(\pi_\infty^W \otimes \pi_f) + m(\bar{\pi}_\infty^H \otimes \pi_f) + m(\bar{\pi}_\infty^W \otimes \pi_f).$

$W_{\pi_f, et}$ Tools

- * HT decomp°
- * Poincaré duality
- * Congruence relation